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Integrals of two confluent hypergeometric functions

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Abstract. The general formula for the integrals containing two confluent hypergeometric functions is obtained in a simple form.

Let us consider integrals of the form

$$J_c^{sp}(a, a') = \int_0^\infty \exp(-hz) z^{c-1+s} F(a, c, kz) F(a', c-p, k'z)\, dz. \tag{1}$$

The values of the parameters are supposed such that the integral converges absolutely; s and p are positive integers, $\mathrm{Re}\, h > 0$.

In Landau and Lifshitz (1959) it is said that the general formula for such integrals can be derived in the way proposed by Gordon (1929), but it is so complex that it cannot be used conveniently. We propose a method of derivation which allows one to obtain the general formula for integrals (1) in a simple form.

Let us use for one of the confluent hypergeometric functions the integral representation (Bateman 1953)

$$F(a, c, kz) = [\Gamma(c)/\Gamma(a)\Gamma(c-a)] \int_0^1 \exp(kzt) t^{a-1}(1-t)^{c-a-1}\, dt. \tag{2}$$

Then we can rewrite the integral (1) in the form

$$J_c^{sp}(a, a') = [\Gamma(c)/\Gamma(a)\Gamma(c-a)] \int_0^1 t^{a-1}(1-t)^{c-a-1} I_c^{sp}(a', t)\, dt \tag{3}$$

where

$$I_c^{sp}(a', t) = \int_0^\infty \exp[-(h-kt)z] z^{c-1+s} F(a', c-p, k'z)\, dz. \tag{4}$$

The integral (4) contains only one hypergeometric function and is easy to evaluate (Landau and Lifshitz 1959)

$$\begin{aligned} I_c^{sp}(a', t) = {} & \Gamma(c+s)(h-kt)^{-c-s}[h/(h-k')]^{a'}[(1-kt/h)/(1-kt/(h-k'))]^{a'} \\ & \times \sum_{m=0}^{s+p} [(a')_m(-s-p)_m(k')^m/(c-p)_m(k'-h)^m(1-kt/(h-k'))^m m!]. \end{aligned} \tag{5}$$

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To calculate the integral (1) we have to integrate over t

$$I_c^s(a, a') = \int_0^1 t^{a-1}(1-t)^{c-a-1}[1-kt/(h-k')]^{-a'-m}(1-kt/h)^{a'-s-c}\,\mathrm{d}t. \qquad (6)$$

This integral is the integral representation for Appell's series F_1 of two variables (Bateman 1953)

$$I_c^s(a, a') = \Gamma(a)\Gamma(c-a)/\Gamma(c)F_1(a, a'+m, s+c-a', c; k/(h-k'), k/h). \qquad (7)$$

From (3), (5) and (7) we obtain the formula for the integral (1) in the form

$$\begin{aligned} J_c^{sp}(a, a') = {} & \Gamma(s+c)h^{-c-s}[h/(h-k')]^{a'} \\ & \times \sum_{m=0}^{s+p} [(a')_m(-s-p)_m(k')^m/(c-p)_m(k'-h)^m m!] \\ & \times F_1(a, c+s-a', a'+m, c; k/h, k/(h-k')). \end{aligned} \qquad (8)$$

There are no restrictions for a, c and $c-a$ due to the gamma functions in (2), (3) and (7) as these functions are in the intermediate steps of the derivation, but do not appear in expression (8). Appell's hypergeometric series F_1 may be expressed as a series in terms of Gauss functions

$$F_1(a, b, b', c; x, y) = \sum_{n=0}^{\infty} [(a)_n(b)_n x^n/(c)_n n!]_2F_1(a+n, b', c+n; y). \qquad (9)$$

Series (9) converges if both variables x and y are less than unity in absolute value. We may transform Appell's series into a series of the same type and obtain the general formula for the integral (1) in the form

$$\begin{aligned} J_c^{sp}(a, a') = {} & \Gamma(c+s)h^{-c-s}(1-k'/h)^{-a'}(1-k/h)^{-a} \\ & \times \sum_{m=0}^{s+p} [(a')_m(-s-p)_m(k')^m/(c-p)_m(k'-h)^m m!] \\ & \times F_1(a, -s-m, a'+m, c; k/(k-h), [kk'/(h-k)(h-k')]). \end{aligned} \qquad (10)$$

As the second parameter of the F_1 function in (10) is a negative integer, we have the integral in terms of Gauss functions

$$\begin{aligned} J_c^{sp}(a, a') = {} & \Gamma(c+s)h^{-c-s}(1-k'/h)^{-a'}(1-k/h)^{-a} \\ & \times \sum_{m=0}^{s+p} [(a')_m(-s-p)_m(k')^m/(c-p)_m(k'-h)^m m!] \\ & \times \sum_{r=0}^{s+m} [(a)_r(-s-m)_r k^r/(c)_r(k-h)^r r!] \\ & \times {}_2F_1(a+r, a'+m, c+r; [kk'/(h-k)(h-k')]). \end{aligned} \qquad (11)$$

Gauss functions have many analytical continuations, which enable one to get convergent series when the variable of the Gauss hypergeometric series in (11) is greater than unity in absolute value.

In the case $s=p=0$ and $h=(k+k')/2$ we have

$$J_c^{00}(a, a') = 2^c\Gamma(c)(k+k')^{a+a'-c}(k'-k)^{-a}(k-k')^{-a'}{}_2F_1(a, a', c; -4kk'/(k'-k)^2). \qquad (12)$$

Equation (12) coincides with (f.13) given by Landau and Lifshitz (1959) for this special case.

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